SURFACES AROUND CLOSED PRINCIPAL CURVATURE LINES, AN INVERSE PROBLEM

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ABSTRACT. Given a non circular spacial closed curve whose total torsion is an integer multiple of 2π, we construct a germ of a smooth surface that contains it as a hyperbolic principal cycle.

1. INTRODUCTION

Let $\alpha : \mathbb{M} \to \mathbb{R}^3$ be a C^r immersion of a smooth, compact and oriented, two–dimensional manifold $\mathbb{M}$ into space $\mathbb{R}^3$ endowed with the canonical inner product $< .,. >$. It will be assumed that $r \geq 4$.

The *Fundamental Forms* of α at a point p of $\mathbb{M}$ are the symmetric bilinear forms on $\mathbb{T}_p\mathbb{M}$ defined as follows [10], [11]:

$$I_\alpha(p; v, w) = \langle D\alpha(p; v), D\alpha(p; w)\rangle \,,$$
$$II_\alpha(p; v, w) = \langle -DN_\alpha(p; v), D\alpha(p; w)\rangle \,.$$

Here, N_α is the positive normal of the immersion α.

The first fundamental form in a local chart (u, v) is defined by $I_\alpha = Edu^2 + 2Fdudv + Gdv^2$, where $E = \langle\alpha_u, \alpha_u\rangle$, $F = \langle\alpha_u, \alpha_v\rangle$ and $G = \langle\alpha_v, \alpha_v\rangle$.

The second fundamental form relative to the unitary normal vector $N_\alpha = (\alpha_u \wedge \alpha_v)/|\alpha_u \wedge \alpha_v|$ is given by $II_\alpha = edu^2 + 2fdudv + gdv^2$, where

$$e = \frac{\det[\alpha_u, \alpha_v, \alpha_{uu}]}{\sqrt{EG - F^2}},\ f = \frac{\det[\alpha_u, \alpha_v, \alpha_{uv}]}{\sqrt{EG - F^2}},\ g = \frac{\det[\alpha_u, \alpha_v, \alpha_{vv}]}{\sqrt{EG - F^2}}.$$

In a local chart (u, v) the principal directions of an immersion α are defined by the implicit differential equation

$$(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2 = 0. \qquad (1)$$

The *umbilic set* of α, denoted by $\mathcal{U}_\alpha$, consists on the points where the three coefficients of equation (1) vanish simultaneously.

The regular integral curves of equation (1) are called *principal curvature lines*. This means curves $c(t) = (u(t), v(t))$, differentiable on an interval, say J, with non–vanishing tangent vector there, such that, for every $t \in J$, it

holds that

$$(Fg - Gf)(u(t), v(t)) \left(\frac{dv(t)}{dt}\right)^2 + (Eg - Ge)(u(t), v(t)) \frac{du(t)}{dt}\frac{dv(t)}{dt} + (Ef - Fe)(u(t), v(t)) \left(\frac{du(t)}{dt}\right)^2 = 0$$

and $c(J) \cap \mathcal{U}_\alpha = \emptyset$.

When the surface $\mathbb{M}$ is oriented, the principal curvature lines on $\mathbb{M} \setminus \mathcal{U}_\alpha$ can be assembled in two one–dimensional orthogonal foliations which will be denoted by $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$. Along the first (resp. second), the normal curvature $II_\alpha(p)$ attains its minimum $k_1(p)$, denominated the *minimal principal curvature at* p, (resp. maximum $k_2(p)$, denominated the *maximal principal curvature at* p).

The triple $\mathcal{P}_\alpha = \{\mathcal{F}_1(\alpha), \mathcal{F}_2(\alpha), \mathcal{U}_\alpha\}$ is called the *principal configuration* of the immersion α, [5], [6]. For a survey about the qualitative theory of principal curvature lines see [2].

A closed principal curvature line, also called a *principal cycle*, is called *hyperbolic* if the first derivative of the Poincaré return map associated to it is different from one.

In [9] and [8] it was proved that a regular closed line of curvature on a surface has as total torsion a multiple of 2π. In this paper we consider the following inverse problem.

Problem 1. Given a simple closed Frenet curve, that is a smooth regular curve of $\mathbb{R}^3$ with non zero curvature, is there an oriented embedded surface that contains it as a hyperbolic principal cycle?

It will be shown that this Problem has a positive answer in the case that the curve is a Frenet, non circular, curve such that its total torsion is an integer multiple of 2π.

The interest of hyperbolic principal cycles is that the asymptotic behavior of the principal foliation around them is determined. The first examples of hyperbolic principal cycles on surfaces were considered by Gutierrez and Sotomayor in [5], where their genericity and structural stability were also established.

2. Preliminary Results

Let $c : [0, L] \to \mathbb{R}^3$ be a smooth *simple, closed, regular curve* in $\mathbb{R}^3$ with positive curvature k and of length $L > 0$, i.e., a Frenet curve. Let also $\mathbf{c} = c([0, L])$. Consider the Frenet frame $\{t, n, b\}$ along $\mathbf{c}$ satisfying the equations

$$\begin{aligned} t'(s) &= k(s)n(s), \\ n'(s) &= -k(s)t(s) + \tau(s)b(s), \\ b'(s) &= -\tau(s)n(s). \end{aligned} \tag{2}$$

Here $k > 0$ is the curvature and τ is the torsion of $\mathbf{c}$.

Consider the parametrized surface of class C^r, $r \geq 4$, defined by the equation

$$\begin{aligned}\alpha(s,v) =& c(s) + [\cos\theta(s) n(s) + \sin\theta(s) b(s)]\, v \\ & + [\cos\theta(s) b(s) - \sin\theta(s) n(s)] \left[\frac{1}{2}A(s)v^2 + \frac{1}{6}B(s)v^3 + v^4 C(s,v)\right] \\ =& c(s) + v(N \wedge T)(s) + \left[\frac{1}{2}A(s)v^2 + \frac{1}{6}B(s)v^3 + v^4 C(s,v)\right] N(s).\end{aligned} \tag{3}$$

For an illustration see Fig. 1 and [4], [5].

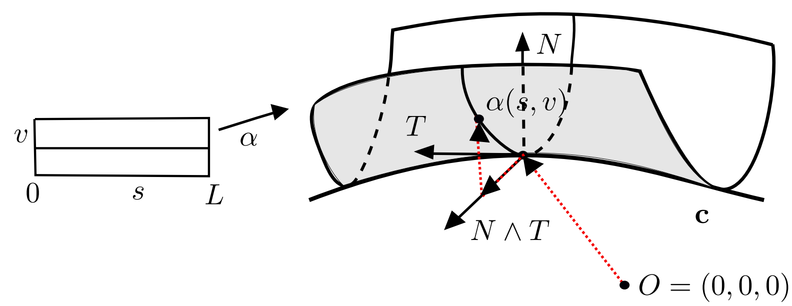

FIGURE 1. Germ of a parametrized surface $\alpha(s,v)$ near a curve $\mathbf{c}$.

Here, $\mathbf{c}'(s) = t(s) = T(s)$, $\theta(s) = \theta(s+L)$, $A(s) = A(s+L)$, $B(s) = B(s+L)$, $C(s,v) = C(s+L,v)$, $C(s,0) = 0$, are smooth L–periodic functions with respect to s and v is small.

Proposition 1. *The curve* $\mathbf{c}$ *is the union of principal curvature lines of* α *if and only if*

$$\tau(s) + \theta'(s) = 0, \;\; \theta(0) = \theta_0, \quad \int_0^L \tau(s) ds = 2m\pi, \quad m \in \mathbb{Z}. \tag{4}$$

Moreover, for any solution $\theta(s)$ *of equation* (4) *the parametric surface defined by equation* (3) *is a regular, oriented and embedded surface in a neighborhood of* $\mathbf{c}$*. The umbilic set* $\mathcal{U}_\alpha \cap \mathbf{c}$ *is defined by the equation*

$$A(s) + k(s)\sin\theta(s) = 0. \tag{5}$$

Proof. We have that $N(s) = N_\alpha(s,0) = \cos\theta(s)b(s) - \sin\theta(s)n(s)$. By Rodrigues formula it follows that $\mathbf{c}$ is a principal curvature line (union of maximal and minimal principal lines) if and only if $N'(s) + \lambda(s)t(s) = 0$. Here λ is a principal curvature (maximal or minimal).

Differentiating N leads to

$$N'(s) = \sin\theta(s)k(s)t(s) - [\tau(s) + \theta'(s)][\cos\theta(s)n(s) + \sin\theta(s)b(s)].$$

Therefore $\mathbf{c}$ is a principal line (union of maximal and minimal principal lines and umbilic points) if and only if $\tau(s) + \theta'(s) = 0$.

By the definition of α it follows that $\alpha_s(s,0) = t(s)$ and $\alpha_v(s,0) = \cos\theta(s)n(s) + \sin\theta(s)b(s)$ are linearly independent and so by the local form of immersions it follows that α is locally a regular surface in a neighborhood of $\mathbf{c}$.

Since the total torsion is an integer multiple of 2π and $\tau(s) + \theta'(s) = 0$ it follows that for any initial condition $\theta(0) = \theta_0$ equation (3) defines an oriented and embedded surface containing $\mathbf{c}$ and having it as the union of principal lines and umbilic points.

Supposing that $\tau(s) + \theta'(s) = 0$ it follows that the coefficients of the first and second fundamental forms of α are given by

$$\begin{aligned}
E(s,v) =& 1 - 2k(s)\cos\theta(s)v \\
&+ \left[\frac{1}{2}k(s)^2(1+\cos 2\theta(s)) + k(s)A(s)\sin\theta(s)\right]v^2 + O(v^3), \\
F(s,v) =& \frac{1}{2}A'(s)A(s)v^3 + O(v^4), \\
G(s,v) =& 1 + A(s)^2v^2 + O(v^3), \\
e(s,v) =& -k(s)\sin\theta(s) + k(s)\cos\theta(s)(\sin\theta(s) - A(s))v + O(v^2), \\
f(s,v) =& A'(s)v + O(v^2), \\
g(s,v) =& A(s) + B(s)v + O(v^2).
\end{aligned} \tag{6}$$

By equations (1) and (6) it follows that the coefficients of the differential equation of principal curvature lines are given by

$$\begin{aligned}
L(s,v) =& (Fg - Gf)(s,v) = -A'(s)v + O(v^2), \\
M(s,v) =& (Eg - Ge)(s,v) = A(s) + k(s)\sin\theta(s) \\
&+ \left[B(s) - k(s)A(s)\cos\theta(s) - \frac{1}{2}k(s)^2\sin 2\theta(s)\right]v + O(v^2), \\
N(s,v) =& (Ef - Fe)(s,v) = A'(s)v + O(v^2).
\end{aligned} \tag{7}$$

The umbilic points along $\mathbf{c}$ are given by the solutions of $M(s,0) = A(s) + k(s)\sin\theta(s) = 0$ which corresponds to the equality between the principal curvatures $k_1(s) = -k(s)\sin\theta(s)$ and $k_2(s) = A(s)$. □

Remark 1. *The one parameter family of surfaces $M(\theta_0) = \alpha_{\theta_0}([0,L] \times (-\epsilon,\epsilon)) \setminus \mathbf{c}$ defined by equations* (3) *and* (4) *is a foliation of a neighborhood of $\mathbf{c}$ after $\mathbf{c}$ is removed. For all θ_0 the curve $\mathbf{c}$ is a principal cycle of α_{θ_0}. This follows from the theorem of existence and uniqueness of ordinary differential equations and smooth dependence with initial conditions of α_{θ_0} and boundary conditions given by equation* (4)*.*

3. Hyperbolic Principal Cycles

In this section it will be given a solution to the problem formulated in the Introduction.

Let α_{θ_0} be the surface defined by equation (3) and associated to the Cauchy problem given by equation (4).

Theorem 2. Consider the oriented parametric surface α_{θ_0} of class C^r, $r \geq 4$, defined by equations (3) and (4) such that $\mathcal{U}_{\alpha_{\theta_0}} \cap \mathbf{c} = \emptyset$. Then $\mathbf{c}$ is a hyperbolic principal cycle of α_{θ_0} if and only if

$$\Lambda(\theta_0) = \int_0^L \frac{A'(s)}{A(s) + k(s)\sin\theta(s)} ds \neq 0. \tag{8}$$

The coefficient $\Lambda(\theta_0)$ is called the characteristic exponent of the Poincaré map associated to $\mathbf{c}$.

Proof. The principal curvatures are given by

$$\begin{aligned} k_1(s,v) =& -k(s)\sin\theta(s) + (k(s)\sin\theta(s) + A(s))k(s)\cos\theta(s)v + O(v^2), \\ k_2(s,v) =& A(s) + B(s)v + O(v^2). \end{aligned}$$

The first return map $\pi : \{s = 0\} \to \{s = L\}$ defined by $\pi(v_0) = v(L, v_0)$, with $v(0, v_0) = v_0$, satisfies the variational equation

$$M(s,0)v_{sv_0}(s) + N_v(s,0)v_{v_0}(s) = 0.$$

By equation (7) it follows that

$$-\frac{N_v}{M}(s,0) = -\frac{A'(s)}{A(s) + k(s)\sin\theta(s)}.$$

Integration of the above equation leads to the result. □

Remark 2. *The criterium of hyperbolicity of a principal cycle was established by Gutierrez and Sotomayor in* [5], [6]*. They proved that a principal cycle* $\mathbf{c}$ *is hyperbolic if and only if*

$$\int_{\mathbf{c}} \frac{dk_1}{k_2 - k_1} = \int_{\mathbf{c}} \frac{dk_2}{k_2 - k_1} = \frac{1}{2}\int_{\mathbf{c}} \frac{d\mathcal{H}}{\sqrt{\mathcal{H}^2 - \mathcal{K}}} \neq 0.$$

Here $\mathcal{H} = (k_1 + k_2)/2$ *and* $\mathcal{K} = k_1 k_2$ *are respectively the* arithmetic mean *and the* Gauss *curvatures of the surface.*

Proposition 2. *Consider the family of oriented parametric surfaces* α_{θ_0} *defined by equations* (3) *and* (4) *such that* $\mathcal{U}_{\alpha_{\theta_0}} \cap \mathbf{c} = \emptyset$ *for all* θ_0*. Then the following holds*

$$\Lambda'(\theta_0) = \int_0^L \frac{k(s)A'(s)\cos(\theta_0 - \int_0^s \tau(s)ds)}{[k(s)\sin(\theta_0 - \int_0^s \tau(s)ds) + A(s)]^2} ds. \tag{9}$$

Proof. Direct differentation of equation (8). □

Theorem 3. Let $\mathbf{c}$ be a smooth curve, that is a closed Frenet curve of length L in $\mathbb{R}^3$ such that τ is not identically zero and $\int_0^L \tau(s)ds = 2m\pi$, $m \in \mathbb{Z}$. Then there exists a germ of an oriented surface of class C^r, $r \geq 4$, containing $\mathbf{c}$ and having it as a hyperbolic principal cycle.

Proof. Consider the parametric surface defined by equation (3). By Proposition 1, $\mathbf{c}$ is a principal cycle when $\theta'(s) = -\tau(s)$, $\theta(0) = \theta_0$, $\int_0^L \tau(s)ds = 2m\pi$. Taking $A(s) = (1 - \sin\theta(s))k(s)$ it follows that $M(s,0) = k(s) > 0$ and so $\mathcal{U}_\alpha \cap \mathbf{c} = \emptyset$. So $\mathbf{c}$ is a closed principal line of the parametric surface α.

By Theorem 2 it follows that $\mathbf{c}$ is hyperbolic if and only if

$$\Lambda = \ln(\pi'(0)) = \int_0^L \frac{A'(s)}{A(s) + k(s)\sin\theta(s)}ds = -\int_0^L \frac{[k(s)\sin\theta(s)]'}{k(s)}ds \neq 0.$$

By assumption the function $k(s)\sin\theta(s)$ is not constant. In fact, $k(s) > 0$ and as τ is not identically equal to zero it follows that $\sin\theta(s) = \sin(\theta_0 - \int_0^s \tau(s)ds)$ is not constant.

If $\Lambda \neq 0$ it follows that $\mathbf{c}$ is hyperbolic and this ends the proof.

In the case when $k(s)\sin\theta(s)$ is not constant and $\int_0^L \frac{[k(s)\sin\theta(s)]'}{k(s)}ds = 0$, consider the deformation of α given by

$$\alpha_\epsilon(s,v) = \alpha(s,v) + \epsilon\frac{a(s)}{2}v^2[\cos\theta(s)b(s) - \sin\theta(s)n(s)],\ a(s) = [k(s)\sin\theta(s)]'.$$

Then $\mathbf{c}$ is a principal cycle of α_ϵ and the principal curvatures are given by

$$\begin{aligned} k_1(s,0,\epsilon) =& -k(s)\sin\theta(s),\\ k_2(s,0,\epsilon) =& A(s) + \epsilon[k(s)\sin\theta(s)]' = (1-\sin\theta(s))k(s) + \epsilon[k(s)\sin\theta(s)]'. \end{aligned}$$

Therefore by Theorem 2 and Remark 2 it follows that

$$\Lambda(\epsilon) = \ln(\pi_\epsilon'(0)) = -\int_0^L \frac{k_1'(s,0,\epsilon)}{k_2(s,0,\epsilon) - k_1(s,0,\epsilon)}ds = \int_0^L \frac{[k(s)\sin\theta(s)]'}{k(s) + \epsilon[k(s)\sin\theta(s)]'}ds.$$

Differentiating the above equation with respect to ϵ and evaluating in $\epsilon = 0$ it follows that

$$\Lambda'(0) = \frac{d}{d\epsilon}\big(\ln(\pi_\epsilon'(0))\big)|_{\epsilon=0} = \int_0^L \left[\frac{[k(s)\sin\theta(s)]'}{k(s)}\right]^2 ds \neq 0.$$

This ends the proof. □

Remark 3. *When the curve* $\mathbf{c}$ *is such that* $\int_0^L \tau(s)ds = 2m\pi,\ m \in \mathbb{Z}\setminus\{0\}$ *there are no ruled surfaces as given by equation* (3) *containing* $\mathbf{c}$ *and having it as a closed principal curvature line. In this situation we have always umbilic points along* $\mathbf{c}$. *In fact, in this case* $k_2(s) = A(s) = 0$ *and* $m \neq 0$ *implies that* $\sin\theta(s)$ *always vanishes. These points, at which* $k_1(s)$ *also vanishes, happen to be the umbilic points. See Proposition 1.*

Corollary 1. *Let* $\mathbf{c}$ *be a closed planar or spherical Frenet curve of length* L *in* $\mathbb{R}^3$. *Then there exists a germ of an oriented surface containing* $\mathbf{c}$ *and having it as a hyperbolic principal cycle if and only if* $\mathbf{c}$ *is not a circle.*

Proof. In the case of a planar curve, let $\mathbf{c}(s) = (x(s), y(s), 0)$ with curvature k and consider the Frenet frame $\{t, n, \mathbf{z}\}$, $\mathbf{z} = (0,0,1)$ associated to $\mathbf{c}$. Any

parametrized surface α containing $\mathbf{c}$ as a principal curvature line has the normal vector equal to $N = \cos\theta(s)n(s) + \sin\theta(s)\mathbf{z}$. Therefore,

$$N' = -k(s)\sin\theta(s)t(s) + \theta'[-\sin\theta(s)\mathbf{z} + \cos\theta(s)n(s)].$$

By Rodrigues formula $N' = -\lambda(s)t(s)$ if and only if $\theta(s) = \theta_0 = cte$. One principal curvature is equal to $k_1(s) = k(s)\sin\theta_0$. By the criterium of hyperbolicity of a principal cycle, see Remark 2 and Theorem 2, the principal curvatures can not be constant along a principal cycle. A construction of the germ of surface containing $\mathbf{c}$ as a hyperbolic principal cycle can be done as in Proposition 1 and Theorem 3. In the case of spherical curves, any closed curve has total torsion equal to zero and the proof follows the same steps of the planar case. This ends the proof. □

4. Concluding Remarks

The study of principal lines goes back to the works of Monge, see [11, page 95], Darboux [1] and many others. In particular, the local behavior of principal lines near umbilic points is a classical subject of research, see [7] for a survey. The structural stability theory and dynamics of principal curvature lines on surfaces was initiated by Gutierrez and Sotomayor [5], [6] and also has been the subject of recent investigation [2].

The possibility of a Frenet (biregular) closed curve in the space to be a principal line of a surface along it depends only on its total torsion to be an integer multiple of 2π.

The presence of umbilic points on such surface depends on function $A(s)$ as well as on $k(s)$ and θ, which in turn depends on τ. In fact $\theta = -\int_0^s \tau(s)ds + \theta_0$, depending on a free parameter θ_0.

In fact the location of the umbilic points given by equation (5) change with the parameter θ_0.

In this paper it has been shown that a non circular closed Frenet curve $\mathbf{c}$ in $\mathbb{R}^3$ can be a principal line of a germ of surface provided its total torsion is $2m\pi,\ m \in \mathbb{Z}$. In Theorem 3 the germ of the surface has been constructed in such a way that $\mathbf{c}$ is a hyperbolic principal line. As it is well known that the total torsion of a closed curve can be any real number, the results of this paper show that closed principal lines (principal cycles) are special curves of $\mathbb{R}^3$. This completes the results established in [8] and [9].

The generic behavior of principal curvature lines near a regular curve of umbilics was studied in [3].

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References

[1] G. Darboux, Leçons sur la Théorie des Surfaces, vols. I, II, III, IV Paris: Gauthier Villars, (1896).

[2] R. Garcia and J. Sotomayor, *Lines of Curvature on Surfaces, Historical Comments and Recent Developments*, São Paulo Journal of Mathematical Sciences, **02**: 99–143, (2008).

[3] R. Garcia and J. Sotomayor, *On the patterns of principal curvature lines around a curve of umbilic points*, Anais Acad. Bras. de Ciências, **77** (2005) 13–24.

[4] R. Garcia and J. Sotomayor, Differential Equations of Classical Geometry, a Qualitative Theory, 27^{th} Brazilian Math. Colloquium, Rio de Janeiro, IMPA, (2009).

[5] C. Gutierrez and J. Sotomayor, *Structural Stable Configurations of Lines of Principal Curvature*, Asterisque, **98–99**: 185–215, (1982).

[6] C. Gutierrez and J. Sotomayor, Lines of Curvature and Umbilic Points on Surfaces, 18^{th} Brazilian Math. Colloquium, Rio de Janeiro, IMPA, 1991. Reprinted as Structurally Configurations of Lines of Curvature and Umbilic Points on Surfaces, Lima, Monografias del IMCA, (1998).

[7] C. Gutierrez and J. Sotomayor, *Lines of curvature, umbilic points and Carathéodory conjecture*, Resenhas IME-USP, **03**: 291–322, (1998).

[8] C.C. Pansonato and S.I.R. Costa, *Total torsion of curves in three-dimensional manifolds*, Geom. Dedicata, **136**: 111–121, (2008).

[9] Yong–An Qin and Shi–Jie Li, *Total torsion of closed lines of curvature*, Bull. Austral. Math. Society, **65**: 73–78, (2002).

[10] M. Spivak, Introduction to Comprehensive Differential Geometry, Vol. III, Berkeley, Publish or Perish, (1999).

[11] D. Struik, Lectures on Classical Differential Geometry, Addison Wesley Pub. Co., Reprinted by Dover Publications, Inc., (1988).

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